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A deformable heat exchanger separated by a helicoid

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Abstract. A heat exchanger consists of a helicoid and the surface of revolution obtained by rotating a profile curve about the axis of the helix. The surfaces are rigidly welded together where they meet. This limits the number of profile curves for which the heat exchanger may be infinitesimally deformed. These profile curves and the deformations they allow are determined by two arbitrary antiperiodic functions.

1. Introduction

A corrugated metal cylinder is often used to allow hot gas to escape from a furnace. The corrugations allow the tube to be bent (into an existing chimney for example). We ask here if it is possible to divide the tube into two by another piece of metal in such a way that the tube can still be bent but act as a heat exchanger.

Consider a helicoid inside a surface of revolution obtained by rotating a curve $g(w)$ about the axis of the helicoid. For practical reasons the two surfaces have to be joined rigidly along the two curves where they meet. So we demand in § 3 that under any deformation, the angles remain unaltered between the triplet of vectors from a point on the curve, one at a tangent to the curve and one other in each surface.

When we attempt to bend the structure so formed it is resisted by forces due to the bending and stretching of the metal. The former is proportional to the distance of a point in the metal from the neutral or middle surface times the change in curvature. For thin metal and small deflections the forces due to bending will be small compared to those due to stretching unless the latter are zero to lowest order in the deflection magnitude. For a general profile curve $g(w)$ it is the resistance to stretching which prevents the structure from being deformed. We shall look for those profile curves which allow deformations in which neither the metal of the helicoid nor of the surface of revolution is stretched. It is sufficient to consider infinitesimal deformations as these already restrict the allowable profile curves. Moreover if terms of higher order in the deformation are to be included, the forces due to bending, which are of comparable order, should also be included, and this is beyond the scope of this paper, though the requirement that large deformations be possible would further restrict the class of profile curves.

A deformation without stretching of the helicoid is given in § 2 by $\varphi(u)$ and $\psi(s)$ functions of the helicoid parameters, a deformation of the surface of revolution by a function $\chi(w, t)$ of its parameters, periodic in t, which satisfies a partial differential equation involving g''/g.

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Section 3 shows that the requirement that, under deformation, common curves in the surfaces remain common and the angles between vectors in the surfaces emanating from a point on the common curve are unchanged, imposes a boundary condition on $\chi(w, t)$ which restricts the possible profile curves $g(w)$ for which the partial differential equation has non-trivial solutions $\chi(w, t)$. The requirement also determines $\varphi(u)$ and $\psi(v)$ from χ and hence the deformation of both surfaces.

We prove in § 3 that not every profile curve $g(w)$ will allow deformations without stretching or tearing. The functional form must be such that g''/g is a continuous twice-differentiable function. The scale of $g(w)$ must be one of a discrete set determined by equation (3.16).

That there are many allowable profile curves $g(w)$ is shown by the numerical method given in the appendix which determines $g(w)$, given arbitrary functions of t: $\chi(0, t)$ and $\chi_w(0, t)$ which change sign under $t \to t+\pi$. The case where these functions are proportional to cos t is solved exactly in § 3. The deformations which these functions $g(w)$ allow are also determined and are consistent with the restrictions.

2. Helicoid inside a surface of revolution

The helicoid can without loss of generality be given parametrically by

$$x(u, s) = ui(s) + sk \tag{2.1}$$

where k is the axis of symmetry and $i(s)$ has components cos s and sin s in the directions perpendicular to k.

The surface of revolution may be written in terms of a continuous function $g(w)$

$$y(w, t) = g(w)i(t) + wk. \tag{2.2}$$

When we bend the structure these surfaces will be deformed into $x+\xi$ and $y+\eta$, say. If no stretching of the metal is to occur then lengths and angles on the surface must be preserved. For infinitesimal ξ, η,

$$x_u \cdot \xi_u = x_s \cdot \xi_s = y_w \cdot \eta_w = y_t \cdot \eta_t = 0 \tag{2.3}$$

and

$$x_u \cdot \xi_s + x_s \cdot \xi_u = y_w \cdot \eta_t + y_t \cdot \eta_w = 0 \tag{2.4}$$

where the subscript denotes the partial derivative with respect to that parameter so that

$$x_u = i(s) \qquad x_s = uj(s) + k \qquad y_w = g'i(t) + k \qquad y_t = gj(t) \tag{2.5}$$

where

$$j(s) = (\mathrm{d}/\mathrm{d}s)i(s). \tag{2.6}$$

Let

$$i \cdot \xi_s = \phi(u, s) \tag{2.7}$$

then from equation (2.4)

$$(uj + k) \cdot \xi_u = -\phi(u, s) \tag{2.8}$$

and from equation (2.3)

$$i \cdot \xi_u = 0 \tag{2.9}$$

$$(uj + k) \cdot \xi_s = 0. \tag{2.10}$$

Taking partial derivatives of equations (2.9) and (2.7) with respect to s and u respectively

$$\boldsymbol{j}\cdot\boldsymbol{\xi}_u=-\boldsymbol{i}\cdot\boldsymbol{\xi}_{us}=-\phi_u \tag{2.11}$$

thus

$$\boldsymbol{\xi}_u=-\phi_u\boldsymbol{j}-(\phi-u\phi_u)\boldsymbol{k}. \tag{2.12}$$

Partially differentiating equation (2.10) with respect to u and equation (2.8) with respect to s we obtain

$$\boldsymbol{j}\cdot\boldsymbol{\xi}_s=-(u\boldsymbol{j}+\boldsymbol{k})\cdot\boldsymbol{\xi}_{su}=\phi_s-u\boldsymbol{i}\cdot\boldsymbol{\xi}_u=\phi_s \tag{2.13}$$

since

$$(\mathrm{d}/\mathrm{d}s)\boldsymbol{j}=-\boldsymbol{i}. \tag{2.14}$$

Thus

$$\boldsymbol{\xi}_s=\phi\boldsymbol{i}+\phi_s(\boldsymbol{j}-u\boldsymbol{k}). \tag{2.15}$$

We have used the $\boldsymbol{i}$ component of the identity of $\boldsymbol{\xi}_{us}$ and $\boldsymbol{\xi}_{su}$ in equation (2.11) and the $(u\boldsymbol{j}+\boldsymbol{k})$ component in equation (2.13). The $(\boldsymbol{j}-u\boldsymbol{k})$ component gives, by partially differentiating equation (2.15) with respect to u and equation (2.12) with respect to s,

$$(1+u^2)\phi_{su}+u\phi_{su}=\boldsymbol{\xi}_{su}\cdot(\boldsymbol{j}-u\boldsymbol{k}) \tag{2.16}$$

$$=-(1+u^2)\phi_{us}+u\phi_s \tag{2.17}$$

thus

$$\phi(u,s)=\varphi(u)+\psi(s) \tag{2.18}$$

$$\boldsymbol{\xi}_u=-\varphi'(\boldsymbol{j}-u\boldsymbol{k})-(\varphi+\psi)\boldsymbol{k}. \tag{2.19}$$

Therefore

$$\boldsymbol{\xi}=-\varphi\boldsymbol{j}+\boldsymbol{k}\int_0^u(u\varphi'-\varphi)\,\mathrm{d}u-u\psi\boldsymbol{k}+\int_0^s(\psi\boldsymbol{i}+\psi'\boldsymbol{j})\,\mathrm{d}s+2(c_1,c_2,c_3) \tag{2.20}$$

since

$$\boldsymbol{\xi}_s=(\varphi+\psi)\boldsymbol{i}+\psi'(\boldsymbol{j}-u\boldsymbol{k}). \tag{2.21}$$

The c_i are constants, allowing for a *relative* translation of the two surfaces. Similarly

$$\boldsymbol{\eta}_w=-g'\chi(w,t)\boldsymbol{j}+g'\chi_t(\boldsymbol{i}-g'\boldsymbol{k}) \tag{2.22}$$

$$\boldsymbol{\eta}_t=g\chi\boldsymbol{i}+gg'(g'')^{-1}\chi_w(\boldsymbol{i}-g'\boldsymbol{k}) \tag{2.23}$$

are the solutions of equations (2.3) and (2.4). They give

$$\boldsymbol{\eta}_{wt}=-g'\chi_t\boldsymbol{j}+g'\chi\boldsymbol{i}+g'\chi_t\boldsymbol{j}+g'\chi_{tt}(\boldsymbol{i}-g'\boldsymbol{k}) \tag{2.24}$$

$$\boldsymbol{\eta}_{tw}=g'\chi\boldsymbol{i}+g\chi_w\boldsymbol{i}+(\boldsymbol{i}-g'\boldsymbol{k})\left(\frac{gg'}{g''}\chi_w\right)_w-gg'\chi_w\boldsymbol{k} \tag{2.25}$$

respectively so that χ may be any function of w and t which satisfies

$$g'\chi_{tt}=\left(\frac{gg'}{g''}\chi_w\right)_w+g\chi_w=-g'\chi+\left(\frac{g}{g''}(g'\chi)_w\right)_w. \tag{2.26}$$

The subscript w indicates the partial derivative of the bracket with respect to w.

Let us now require that the surface of revolution remains closed under the deformation. That is, $\chi(w, t)$ is periodic in t with period 2π. So if

$$2P(w, t)\equiv g'(w)[\chi(w, t)-\chi(w, t+\pi)]=-2P(w, t+\pi) \tag{2.27}$$

$$2Q(w, t)\equiv g'(w)[\chi(w, t)+\chi(w, t+\pi)]=2Q(w, t+\pi) \tag{2.28}$$

then equation (2.26) gives

$$P_{tt}+P=[(g/g'')P_w]_w \tag{2.29}$$

$$Q_{tt}+Q=[(g/g'')Q_w]_w \tag{2.30}$$

allowing us to write

$$P(w, t)=\sum h'_\nu(w)\exp[(2\nu+1)\mathrm{i}t] \qquad h_{-\nu-1}(w)=h^*_\nu(w) \tag{2.31}$$

$$Q(w, t)=\sum f'_\nu(w)\exp(2\nu\mathrm{i}t) \qquad f_{-\nu}(w)=f^*_\nu(w) \tag{2.32}$$

where Σ denotes a sum over integer ν from $-\infty$ to $+\infty$ and

$$h''_\nu=\nu(\nu+1)\mu h_\nu \qquad \nu\neq 0, -1 \tag{2.33}$$

$$h''_0=\alpha\,\mathrm{e}^{\mathrm{i}\theta}\mu \qquad h''_{-1}=\alpha\,\mathrm{e}^{-\mathrm{i}\theta}\mu \qquad \alpha, \theta \text{ constant} \tag{2.34}$$

$$f''_\nu=(\nu^2-\tfrac{1}{4})\mu f_\nu \tag{2.35}$$

where $\mu(w)$ is given in terms of g by

$$4g''+\mu g=0 \tag{2.36}$$

and equations (2.22) and (2.23) give

$$\boldsymbol{\eta}_w(w, t)=-(P+Q)\boldsymbol{j}(t)+(P_t+Q_t)[\boldsymbol{i}(t)-g'(w)\boldsymbol{k}] \tag{2.37}$$

$$\boldsymbol{\eta}_t(w, t)=(P+Q)g\boldsymbol{k}+(g/g'')(P_w+Q_w)[\boldsymbol{i}(t)-g'(w)\boldsymbol{k}] \tag{2.38}$$

$$\boldsymbol{\eta}_w(w, t+\pi)=-(P-Q)\boldsymbol{j}(t)+(P_t-Q_t)[\boldsymbol{i}(t)+g'(w)\boldsymbol{k}] \tag{2.39}$$

$$\boldsymbol{\eta}_t(w, t+\pi)=-(P-Q)g\boldsymbol{k}+(g/g'')(P_w-Q_w)[\boldsymbol{i}(t)+g'(w)\boldsymbol{k}]. \tag{2.40}$$

In § 3 we will find boundary conditions to go with equations (2.29) and (2.30) which will determine P and Q.

3. The common curve

In § 2 we treated the helicoid and the surface of revolution as separate surfaces and only demanded that they do not stretch or tear when the structure is bent. Initially the curves $\boldsymbol{x}(g(v), v)$ and $\boldsymbol{y}(v, v)$ are coincident and the curves $\boldsymbol{x}(-g(v), v)$ and $\boldsymbol{y}(v, v+\pi)$ are coincident. We require that they remain common curves under the deformation in order that the surfaces remain joined together. But we cannot allow this join to be flexible as in practice it has to be welded. Thus we shall require that all tangent vectors to the two surfaces which have their origin at a particular point on a common curve transform rigidly together, i.e. the angle between any two of them remains unchanged under the deformation.

If the coincident curves are to remain common curves the equations

$$\boldsymbol{\xi}(g(v), v) = \boldsymbol{\eta}(v, v) \tag{3.1}$$

$$\boldsymbol{\xi}(-g(v), v) = \boldsymbol{\eta}(v, v+\pi) \tag{3.2}$$

must be satisfied. These give

$$g'\boldsymbol{\xi}_u(g(v), v) + \boldsymbol{\xi}_s(g(v), v) = \boldsymbol{\eta}_w(v, v) + \boldsymbol{\eta}_t(v, v) \tag{3.3}$$

$$-g'\boldsymbol{\xi}_u(-g(v), v) + \boldsymbol{\xi}_s(-g(v), v) = \boldsymbol{\eta}_w(v, v+\pi) + \boldsymbol{\eta}_t(v, v+\pi). \tag{3.4}$$

Using (2.19), (2.21) and (2.37)-(2.40) the $\boldsymbol{j}$ component of equations (3.3) and (3.4) gives

$$p'_+(v) - \psi'(v) = P(v, v) \tag{3.5}$$

$$p'_-(v) = Q(v, v) \tag{3.6}$$

where we have written

$$2p_\pm(v) \equiv \varphi(g(v)) \pm \varphi(-g(v)) \tag{3.7}$$

and the $\boldsymbol{i}$ component gives

$$p_+(v) + \psi(v) = P_t(v, v) + (g/g'')P_w(v, v) \tag{3.8}$$

$$p_-(v) = Q_t(v, v) + (g/g'')Q_w(v, v). \tag{3.9}$$

The $\boldsymbol{k}$ component is then also satisfied.

We next demand that the angle between a tangent vector in the helicoid and a tangent vector in the surface of revolution, both vectors originating from the same point on a common curve, be preserved under the deformation

$$\boldsymbol{x}_u(g(v), v) \cdot \boldsymbol{\eta}_w(v, v) + \boldsymbol{y}_w(v, v) \cdot \boldsymbol{\xi}_u(g(v), v) = 0 \tag{3.10}$$

$$\boldsymbol{x}_u(-g(v), v) \cdot \boldsymbol{\eta}_w(v, v+\pi) + \boldsymbol{y}_w(v, v+\pi) \cdot \boldsymbol{\xi}_u(-g(v), v) = 0 \tag{3.11}$$

which using equations (2.5), (2.19), (2.21), (3.7) and (2.37)-(2.40) gives

$$P_t - (p_+ + \psi) + (g/g')p'_+ = 0 \qquad \text{when } w = t \tag{3.12}$$

$$Q_t - p_- + (g/g')p'_- = 0 \qquad \text{when } w = t. \tag{3.13}$$

Equations (2.32), (2.30), (3.6), (3.9) and (3.13) determine $f_\nu(w)$ and hence Q and p_-

$$f_\nu(w) \equiv 0 \qquad \nu \neq 0 \qquad f_0(w) = \theta g(w) \tag{3.14}$$

$$p_-(w) = \theta g(w) \qquad Q(w, t) = \theta g'(w). \tag{3.15}$$

This value of $Q(w, t)$ does not correspond to a true deformation but to a rotation of the system through a constant angle θ about the $\boldsymbol{k}$ axis.

Two of the equations (3.5), (3.8) and (3.12) may be regarded as defining $p_+(w)$ and $\psi(w)$ and the third gives the boundary condition for equation (2.29)

$$0 = \sum{}' \left[\left(\frac{g'}{g} - \frac{h'_\nu}{h_\nu}\right) + \frac{\mathrm{i}}{8}(2\nu+1)(\mu-4)\right] h_\nu \exp[(2\nu+1)\mathrm{i}w]$$
$$+ 2\alpha\left(\frac{g'}{g}\cos(w+\theta) - \tfrac{1}{8}(\mu-4)\sin(w+\theta)\right) \tag{3.16}$$

where Σ' denotes the sum over ν from $-\infty$ to $+\infty$ excluding $\nu=0,-1$, and h_ν is given by equation (2.33).

The trivial solution $h_\nu \equiv 0$, $\nu \neq 0, -1$ and $\alpha=0$ which gives

$$P=\beta\cos(t+\theta) \qquad p_+(v)\equiv 0 \qquad \psi=-\beta\sin(v+\theta) \tag{3.17}$$

corresponds to rotation of the whole system through a constant angle β about a fixed axis $\boldsymbol{i}(\theta)$ perpendicular to $\boldsymbol{k}$.

There are no solutions giving non-trivial deformations for arbitrary functions $g(w)$. For a given functional form $\mu(w)=-\lambda^2 f^2(w)$, equation (3.16) with $h_\nu(w)$ solutions of equation (2.33) may be considered as an eigenvalue equation for the eigenvector $\{\alpha\,\mathrm{e}^{\pm\mathrm{i}\theta}, h_\nu^{(0)}, h_\nu'^{(0)}\}$ and eigenvalue λ.

As an example let us consider the functions f which satisfy

$$2\left(\frac{f'}{f}\right)_w-\left(\frac{f'}{f}\right)^2=0. \tag{3.18}$$

Then

$$g(w)=\frac{a}{\sqrt{f}}\cosh\tfrac{1}{2}\lambda\int f\,\mathrm{d}w \tag{3.19}$$

$$h_\nu(w)=\frac{\alpha_\nu}{\sqrt{f}}\cos\left(\lambda[\nu(\nu+1)]^{1/2}\int f\,\mathrm{d}w\right) \qquad \nu\neq 0,-1 \tag{3.20}$$

satisfy (2.36) and (2.33) respectively with $\mu=-\lambda^2 f^2(w)$. Equation (3.16) becomes

$$\begin{aligned}\sum{}'\alpha_\nu\Bigg\{\lambda f\left[\tanh\left(\tfrac{1}{2}\lambda\int f\right)+2[\nu(\nu+1)]^{1/2}\tan\left(\lambda[\nu(\nu+1)]^{1/2}\int f\,\mathrm{d}w\right)\right]\\ -\tfrac{1}{4}\mathrm{i}(2\nu+1)(\lambda^2 f^2+4)\Bigg\}\exp[(2\nu+1)\mathrm{i}w]\cos\left([\nu(\nu+1)]^{1/2}\lambda\int f\right)\\ -\left\{2\left[f'-\lambda f^2\tanh\left(\tfrac{1}{2}\lambda\int f\right)\right]\cos(w+\theta)\right.\\ \left.-\tfrac{1}{2}\left(\lambda^2 f^2+4\right)f\sin(w+\theta)\right\}f^{-1/2}\alpha=0\end{aligned} \tag{3.21}$$

an eigenvalue equation for λ.

$$\sum A_{\sigma\nu}(\lambda)\alpha_\nu=0 \tag{3.22}$$

$$A_{\sigma,-\nu-1}=A_{\sigma\nu}^* \qquad \alpha_{-\nu-1}=\alpha_\nu^* \tag{3.23}$$

where $A_{\sigma\nu}(\lambda)$ is the coefficient of w^σ in

$$\begin{aligned}(\lambda\tanh(\beta-\tfrac{1}{2}\lambda w)+2\lambda[\nu(\nu+1)]^{1/2}\tan\{\beta_\nu-\lambda[\nu(\nu+1)]^{1/2}w\}+\tfrac{1}{4}\mathrm{i}(2\nu+1)(\lambda^2+4))\\ \times\exp[(2\nu+1)\mathrm{i}w]\cos\{\beta_\nu-\lambda[\nu(\nu+1)]^{1/2}w\}\end{aligned} \tag{3.24}$$

for the constant solutions of (3.18). Thus the profile curve

$$g(w)=a\cosh(\beta-\tfrac{1}{2}\lambda w) \tag{3.25}$$

is permissible with any of these eigenvalues.

The deformation $\boldsymbol{\eta}(wt)$, $\boldsymbol{\xi}(u, v)$ is obtained from equations (2.20), (2.22) and (2.23) using

$$\chi(w,t) = -2\Big(\sum{}' \alpha_\nu[\nu(\nu+1)]^{1/2}\exp[(2\nu+1)it]\sin\{\beta_\nu - \lambda[\nu(\nu+1)]^{1/2}w\} - 2\lambda w\alpha\cos(t+\theta)\Big)[a\sinh(\beta-\tfrac{1}{2}\lambda w)]^{-1} \tag{3.26}$$

$$\varphi'[a\cosh(\beta-\tfrac{1}{2}\lambda w)] - \psi'(w)[a\lambda\sinh(\beta-\tfrac{1}{2}\lambda w)]^{-1} = \chi(w,w) \tag{3.27}$$

$$\begin{aligned}\varphi[a\cosh(\beta-\tfrac{1}{2}\lambda w)] + \psi(w) = \sum{}' &((2\nu+1)i\lambda[\nu(\nu+1)]^{1/2}\sin\{\beta_\nu - [\nu(\nu+1)]^{1/2}w\}\\ &-4\nu(\nu+1)\cos\{\beta_\nu-\lambda[\nu(\nu+1)]^{1/2}w\})\alpha_\nu\exp[(2\nu+1)iw]\\ &+2\alpha\lambda^2 w\sin(w+\theta)+8\alpha\cos(w+\theta).\end{aligned} \tag{3.28}$$

The other solutions of equation (3.18)

$$f = (w+c)^{-2} \tag{3.29}$$

give a profile curve

$$g(w) = a(w+c)\cosh[\beta-\tfrac{1}{2}\lambda(w+c)^{-1}] \tag{3.30}$$

which is permissible for eigenvalues λ of an eigenvalue equation obtained in a similar manner. The deformation is determined by the eigenvector $\{\alpha\,e^{\pm i\theta}, \alpha_\nu, \beta_\nu\}$ since it is given in terms of $\chi(w, t)$, $\phi(u)$, $\psi(v)$ where

$$\begin{aligned}\chi(w,t) = \Big(\sum{}' &((w+c)\cos\{\beta_\nu-\lambda[\nu(\nu+1)]^{1/2}(w+c)^{-1}\}\\ &-\lambda[\nu(\nu+1)]^{1/2}\sin\{\beta_\nu-\lambda[\nu(\nu+1)]^{1/2}(w+c)^{-1}\})\\ &\times\exp[(2\nu+1)it]\alpha_\nu + \tfrac{2}{3}\lambda^2\alpha\cos(t+\theta)(w+c)^{-2}\Big)\\ &\times(a\{(w+c)\cosh[\beta-\tfrac{1}{2}\lambda(w+c)^{-1}]+\tfrac{1}{2}\lambda\sinh[\beta-\tfrac{1}{2}\lambda(w+c)^{-1}]\})^{-1}\end{aligned} \tag{3.31}$$

$$\begin{aligned}\varphi'\{a(w+c)\cosh[\beta-\tfrac{1}{2}\lambda(w+c)^{-1}]\} - \psi'(w)\{&a\cosh[\beta-\tfrac{1}{2}\lambda(w+c)^{-1}]\\ &+\tfrac{1}{2}\lambda(w+c)^{-1}\sinh[\beta-\tfrac{1}{2}\lambda(w+c)^{-1}]\}^{-1} = \chi(w,w)\end{aligned} \tag{3.32}$$

$$\begin{aligned}\varphi\{a(w+c)&\cosh[\beta-\tfrac{1}{2}\lambda(w+c)^{-1}]\} - \psi(w)\\ &=\sum{}'([(2\nu+1)i-4\nu(\nu+1)(w+c)]\cos\{\beta_\nu-\lambda[\nu(\nu+1)]^{1/2}(w+c)^{-1}\}\\ &\quad-(2\nu+1)i(w+c)^{-1}\lambda[\nu(\nu+1)]^{1/2}\sin\{\beta_\nu-\lambda[\nu(\nu+1)]^{1/2}(w+c)^{-1}\})\\ &\quad\times\exp[(2\nu+1)iw]\\ &\quad-\tfrac{2}{3}\lambda^2\alpha(w+c)^{-3}\sin(t+\theta)-8\alpha\cos(t+\theta).\end{aligned} \tag{3.33}$$

In general the permissible profile curves $g(w)$ may be classified as follows. $g(w)$ satisfies

$$g'' - (\tfrac{1}{2}\lambda)^2 f^2(w) g = 0 \tag{3.34}$$

where $f(w)$ is an arbitrary continuous function but λ must satisfy an eigenvalue equation (3.16). The deformations of the helicoid joined to the surface of revolution with this profile curve is given by the eigenvector and will in general involve undesirably high Fourier modes in the variable t which may make the radius of curvature too small in places for us to neglect the forces due to bending.

There is however an alternative classification of profile curves $g(w)$. Given α, θ, $h_\nu(0)$, $h'_\nu(0)$, i.e. $\chi(0, t)$, $\chi_w(0, t)$ and $g(0)$, $g'(0)$, equations (3.16), (2.33), (2.34) and (2.36) determine $g(w)$ uniquely. The numerical method of solution given in the appendix serves as a proof of this statement.

There is however the following example which can be treated analytically.

Take $h_\nu(0) = h'_\nu(0) = 0$ so that $h_\nu(0) \equiv 0$, $\nu \neq 0, -1$, then equations (2.36) and (3.16) give

$$g'' + 2g' \cot w + g = 0 \tag{3.35}$$

where we have taken $\theta = 0$ without loss of generality:

$$g = c\frac{\cos(\sqrt{2}\, w + \beta)}{\sin w} \qquad 0 < w < \pi \tag{3.36}$$

and

$$P(w, t) = 2\alpha M' \cos t \tag{3.37}$$

$$\boldsymbol{\eta}(w, t) = 2\alpha[-2t - \sin 2t, \cos 2t - M, (M'g + 4g') \sin t] \tag{3.38}$$

where

$$M'' = \mu = 4(g + 2g' \cot w)/g \tag{3.39}$$

$$\varphi(u) = -8\alpha \int u^{-1} \cos v(u)\, \mathrm{d}u \tag{3.40}$$

where $v(u)$ is defined with range $0 < v < \pi$ by

$$u = c\frac{\cos(\sqrt{2}\, v + \beta)}{\sin v} \tag{3.41}$$

$$\psi(w) = -2\alpha \int (M' + 4g'/g) \cos w \, \mathrm{d}w. \tag{3.42}$$

The distorted system is given by

$$u\boldsymbol{i}(s) - \varphi(u)\boldsymbol{j}(s) + \boldsymbol{k}\left(s + \int (u\varphi' - \varphi)\, \mathrm{d}u - u\psi(s)\right) + \int_0^s (\psi\boldsymbol{i} + \psi'\boldsymbol{j})\, \mathrm{d}s \tag{3.43}$$

and

$$[g(w) \cos t - 2t - \sin 2t, g(w) \sin t + \cos 2t - M, w + (M'g + 4g') \sin t] \tag{3.44}$$

joined rigidly together along $u = g(s)$ on one surface and $w = t$ on the other; also along $u = -g(s)$ and $w + \pi = t$.

The curvature of these surfaces is such that bending strains remain small for thin metal. Unfortunately $g(w)$ does not remain finite for a full pitch of the helicoid.

If $h_\nu(0)$, $h'_\nu(0)$ are small compared to α the solution $g(w)$, $\mu(w)$ can be determined from the case above by perturbation theory.

If $h_\nu(0)$, $h'_\nu(0)$, α, θ, $g(0)$, $g'(0)$ determine a smooth function $\mu(w)$ (which we write as $-\lambda^2 f^2(w)$) the solution is given in the WKB approximation by equations (3.19) and (3.20) where f is a solution of the second-order differential equation (3.21).

If $\alpha = 0$ equation (3.21) is first order and may be written in terms of $y \equiv \frac{1}{2}\lambda \int f$ as

$$dy/dw = F \pm (F^2 - 1)^{1/2} \tag{3.45}$$

where

$$F(w, y) = \left(\sum_{\nu=1}^{\infty} |\alpha_\nu| (\tanh y + 2[\nu(\nu+1)]^{1/2} \tan\{2[\nu(\nu+1)]^{1/2} y + \beta_\nu\}) \times \cos[(2\nu+1)w] \cos\{2[\nu(\nu+1)]^{1/2} y + \beta_\nu\} \right) \times \left(\sum_{\nu=1}^{\infty} |\alpha_\nu| (2\nu+1) \sin[(2\nu+1)w] \cos\{2[\nu(\nu+1)]^{1/2} y + \beta_\nu\} \right)^{-1}. \tag{3.46}$$

For consistency of the WKB approximation the resulting solution must be so smooth that f'/f remains small, i.e. $(f'/f)^2$ and $(d/dw)(f'/f)$ are both negligible so that equation (3.8) is approximately satisfied. This will only be true for certain values of $|\alpha_\nu|$.

The deformation is given by $\chi(w, t)$, $\varphi(u)$, $\psi(v)$ where

$$\chi(w, t) = \sum{}' (4[\nu(\nu+1)]^{1/2} (y')^2 \sin\{2[\nu(\nu+1)]^{1/2} y + \beta_\nu\} + y'' \cos\{2[\nu(\nu+1)]^{1/2} y + \beta_\nu\}) \times |\alpha_\nu| \cos[(2\nu+1)t][\tfrac{1}{2}a(y'' \cosh y - 2y' \sinh y)]^{-1} \tag{3.47}$$

$$\varphi'(a\sqrt{\lambda} \cosh y/\sqrt{2y'}) + 2(y')^{3/2} \psi'(w)[a(y'' \cosh y - 2y' \sinh y)]^{-1} = \chi(w, w) \tag{3.48}$$

$$\varphi(a\sqrt{\lambda} \cosh y/\sqrt{2y'}) + \psi(w) = \sum [(2\nu+1)(4[\nu(\nu+1)]^{1/2}(y')^2 \sin\{2[\nu(\nu+1)]^{1/2} y + \beta_\nu\} + y'' \cos\{2[\nu(\nu+1)]^{1/2} y + \beta_\nu\}) \sin[(2\nu+1)w] - 8\nu(\nu+1) y' \cos\{2[\nu(\nu+1)]^{1/2} y + \beta_\nu\} \times \cos[(2\nu+1)w]\alpha_\nu (2y')^{-3/2}. \tag{3.49}$$

The deformation given by inserting φ, ψ and χ into equations (2.20), (2.22) and (2.23) is smooth and does not have large curvature if we take $|\alpha_\nu|$ zero for larger values of ν.

4. Conclusions

A heat exchanger consists of a helicoid inside a surface of revolution obtained by rotating a curve $g(w)$ about the axis of the helicoid. The two surfaces are welded inflexibly along the common curve.

We have shown that this system is in general rigid for an arbitrary profile curve $g(w)$, but that a large class of profile curves $g(w)$ allow a deformation without stretching and these $g(w)$, together with $\phi(u)$, $\psi(v)$ and $\chi(w, t)$ which give the deformation, are determined by two periodic functions $\chi(0, t)$, $\chi_w(0, t)$ (which change sign when $t \to t + \pi$) together with $g(0)$, $g'(0)$. If $\chi(0, t)$, $\chi_w(0, t)$ contain high Fourier components, bending stress in the metal will keep the system rigid.

We have given an analytic solution $g(w)$ and the deformed structure when $\chi(0, t)$, $\chi_w(0, t)$ are proportional to $\cos t$. Other solutions have to be obtained numerically, cf the appendix. However we are mainly interested in finding allowable profile curves $g(w)$ rather than writing down the deformation it permits. In particular we would like to have found a $g(w)$ with period 2π. This is very restrictive since it requires that $\mu(w)$ in equation (2.36) has this period and to have a scale which allows a periodic solution. This will not be compatible with the scale being an eigenvalue of (3.16) for arbitrary periodic functions $\mu(w)$. We must embark on a search, changing $\chi(0, t)$, $\chi_w(0, t)$.

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Appendix

We give here an algorithm to solve the equations (2.33) and (2.36) when μ is given by (3.16). This is equivalent to finding the solution of the partial differential equation (2.29) from the initial values $P(0, t)$, $P_w(0, t)$ when $g(w)$, $\psi(w)$ and $p_+(w)$ are given by equations (3.6), (3.8) and (3.12).

Let

$$\eta_{\nu,n+1}=2\eta_{\nu,n}-\eta_{\nu,n-1}+\varepsilon^2\nu(\nu+1)\mu_n h_{\nu,n} \tag{A1}$$

$$\eta_{n+1}=2\eta_n-\eta_{n-1}-\tfrac{1}{4}\varepsilon^2\mu_n g_n. \tag{A2}$$

Then μ is given by

$$\frac{(\mu_{n+1}-4)T_{n+1}}{1+\frac{1}{48}\varepsilon^2\mu_{n+1}}=-\frac{2\varepsilon}{3}\left(\frac{\mu_{n+1}\eta_{n+1}S_{n+1}}{1+\frac{1}{48}\varepsilon^2\mu_{n+1}}+\frac{4\mu_n\eta_n S_n}{1+\frac{1}{48}\varepsilon^2\mu_{n+1}}\right)+V_{n-1} \tag{A3}$$

where

$$T_n=\mathrm{i}\sum\frac{(2\nu+1)\exp[(2\nu+1)\mathrm{i}n\varepsilon]\eta_{\nu,n}}{1-\frac{1}{12}\varepsilon^2\nu(\nu+1)\mu_n} \tag{A4}$$

$$S_n=\sum\frac{(2\nu+1)^2\exp[(2\nu+1)\mathrm{i}n\varepsilon]\eta_{\nu,n}}{1-\frac{1}{12}\varepsilon^2\nu(\nu+1)\mu_n} \tag{A5}$$

$$V_n=\frac{\eta_n}{1-\frac{1}{48}\varepsilon^2\mu_n}[-\tfrac{2}{3}\varepsilon\mu_n S_n+(\mu_n-4)T_n] \tag{A6}$$

and h and g are

$$h_{\nu,n+1}=\eta_{\nu,n+1}[1-\tfrac{1}{12}\varepsilon^2\nu(\nu+1)\mu_{n+1}]^{-1} \tag{A7}$$

$$g_{n+1}=\eta_{n+1}[1+\tfrac{1}{48}\varepsilon^2\mu_{n+1}]^{-1}. \tag{A8}$$

Equation (A3) uses Simpson's rule with error $O(\varepsilon^4)$ in equation (3.16) written in the form

$$(\mu-4)\mathrm{i}g\sum_{\nu=-\infty}^{+\infty}(2\nu+1)\exp[(2\nu+1)\mathrm{i}w]h_\nu$$
$$=2\sum_{\nu=-\infty}^{+\infty}(2\nu+1)^2\exp[(2\nu+1)\mathrm{i}w]\int\mu g h_\nu\,\mathrm{d}w \tag{A9}$$

where for notational convenience h_0, h_{-1} shall be the constants $\alpha\,e^{i\theta}$, $\alpha\,e^{-i\theta}$ in this appendix rather than having the meaning they have in the text given by equation (2.34).

Equation (A3) is an algebraic equation for μ_{n+1}, all other terms being given at a previous stage in the iteration. The error involved in an iterative solution of this algebraic equation can be made smaller than $O(\varepsilon^4)$.

Finally the error involved in the Numerov part of the iteration, equations (A1) and (A7) being the solution of equation (2.33) given μ and equations (A2) and (A8) being the solution of (2.36) given μ, is $O(\varepsilon^6)$.

Given $h_\nu(0)$, $h'_\nu(0)$, $g(0)$, $g'(0)$ we can calculate $\mu(0)$, $\mu'(0)$ and $\mu''(0)$ from (3.16) and its derivative at $w=0$ and thus $\eta_{\nu,0}$ and η_0 from (A7) and (A8):

$$g_1 = g(0)+g'(0)-\tfrac{1}{8}\mu(0)g(0)\varepsilon^2-\tfrac{1}{24}(\mu'(0)g(0)-\mu(0)g'(0))\varepsilon^3+(1/4!)g^{(4)}(0)\varepsilon^4$$

$$h_{\nu 1} = h_\nu(0)+h'_\nu(0)+\tfrac{1}{2}\nu(\nu+1)\mu(0)h_\nu(0)\varepsilon^2$$
$$+\tfrac{1}{6}\nu(\nu+1)[\mu'(0)h_\nu(0)-\mu(0)h'_\nu(0)]\varepsilon^3+(1/4!)h_\nu^{(4)}(0)\varepsilon^4.$$

The ε^4 terms needed to calculate μ_1 from equation (3.16) at $w=\varepsilon$ to $O(\varepsilon^4)$ occur in the combination

$$g^{(4)}(0)h_\nu(0)-g(0)h_\nu^{(4)}(0)$$
$$=[-\tfrac{1}{4}(2\nu+1)^2\mu''(0)+\tfrac{1}{16}\mu^2(0)-\nu^2(\nu+1)^2]g(0)h(0)$$
$$-\tfrac{1}{2}\mu'(0)g'(0)h_\nu(0)-2\nu(\nu+1)\mu'(0)g(0)h'_\nu(0)$$

which is determined since we know $\mu''(0)$ from (3.16). Knowing μ_1, we can calculate η_1 and $\eta_{\nu 1}$ from g_1, $h_{\nu 1}$ using equations (A7) and (A8) and set the iteration going at (A1).